Mittag-Leffer's theorem. Laurent series. Partial fractions expansions

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Remark about two-sided series

Def $\sum_{n=-\infty}^{\infty} b_n$ converges if $\exists \lim_{N_1, N_2 \to \infty} \sum_{n=-N_1}^{N_2} b_n \iff$ both $\sum_{n=0}^{\infty} b_n$ and $\sum_{n=1}^{\infty} b_{-n}$ converge.

If $\sum_{n=0}^{\infty} a_n w^n$ a power series with radius of convergence R,

then $\sum_{n=0}^{\infty} \frac{a_n}{(z-z_0)^n}$ converges locally uniformly when $\frac{1}{|z-z_0|} < R \iff |z-z_0| > \frac{1}{R}$.

$f(z) := \sum_{n=0}^{\infty} \frac{a_n}{(z-z_0)^n} \in A\left(\{|z-z_0| > \frac{1}{R}\}\right)$ including ∞: $f(\infty) = a_0$.
removable singularity.

Let now $(a_n)_{n=-\infty}^{\infty}$, then

$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ converges in some annulus $\{R_1 < |z-z_0| < R_2\}$.
where R_2 - radius of convergence of $\sum_{n=0}^{\infty} a_n w^n$
$1/R_1$ - radius of convergence of $\sum_{n=1}^{\infty} a_{-n} w^n$.

$f \in A(\{R_1 < |z-z_0| < R_2\})$ — locally uniform sum.

Note. Possible that $R_1 = 0$, $R_2 = \infty$.

Pierre Alphonse Laurent

Theorem (Laurent) Let $f \in A(\{R_1 < |z-z_0| < R_2\}) =: \Omega$

Then $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$.

The series converges locally uniformly in Ω.

$a_n = \frac{1}{2\pi i} \oint_{C_r} f(\zeta)(\zeta - z_0)^{-(n+1)} d\zeta$ for any $R_1 < r < R_2$,
$C_r := \{z + re^{it}\}$,

Proof

Observe: $\forall\ R_1 < r_1 < r_2 < R_2$, $\quad C_{r_1} - C_{r_2} \sim 0$ in Ω.

Indeed, $z \notin \Omega \Rightarrow$ $|z| \geq R_2$: $n(C_{r_1}, z) = n(C_{r_2}, z) = 0$

$|z| \leq R_1$: $n(C_{r_1}, z) = n(C_{r_2}, z) = 1$.

So $\oint_{C_{r_1}} f(\zeta)(\zeta - z_0)^n d\zeta = \oint_{C_{r_2}} f(\zeta)(\zeta - z_0)^n d\zeta \quad \forall n \in \mathbb{Z}$ (can be negative).

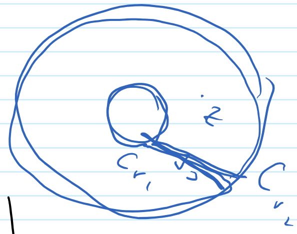
since $f(\zeta)(\zeta - z_0)^n \in A(\Omega)$.

Let us fix $z \in \Omega$ and r_1, r_2: $R_1 < r_1 < |z - z_0| < r_2 < R_2$

Notice $n(C_{r_2} - C_{r_1}, z) = 1$ $(n(C_{r_1}, z) = 0,\ n(C_{r_2}, z) = 1)$.

By Cauchy integral formula:

$$f(z) = \frac{1}{2\pi i} \oint_{C_{r_2}} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \oint_{C_{r_1}} \frac{f(\zeta)}{\zeta - z} d\zeta$$



Use Cauchy trick twice:

On $|\zeta - z_0| = r_2 > |z - z_0|$:

$$\frac{1}{\zeta - z} = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}} \quad \left(\frac{|z - z_0|}{|\zeta - z_0|} < 1 \right)$$ – converges uniformly. On C_{r_2}

On $|\zeta - z_0| = r_1 < |z - z_0|$

$$-\frac{1}{\zeta - z} = \frac{1}{(z - z_0)\left(1 - \frac{\zeta - z_0}{z - z_0}\right)} = \sum_{k=1}^{\infty} \frac{(\zeta - z_0)^{k-1}}{(z - z_0)^k}$$ – converges uniformly on C_{r_1}.

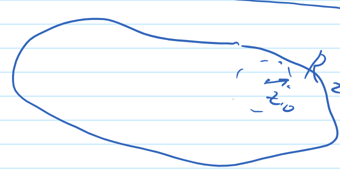
Since multiplication by bounded $f(z)$ does not change uniform convergence, we get

$$f(z) = \sum_{k=0}^{\infty} \underbrace{\frac{1}{2\pi i} \oint_{C_{r_2}} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta}_{a_k} (z - z_0)^k + \sum_{k=1}^{\infty} \underbrace{\frac{1}{2\pi i} \oint_{C_{r_1}} f(\zeta)(\zeta - z_0)^{k-1} d\zeta}_{a_{-k}} (z - z_0)^{-k}$$

The series converges for every $z \in \Omega$, so locally uniformly ∎

Special case: $R_1 = 0$ - isolated singularity.

If $f \in A(\Omega \setminus \{z_0\})$, $\quad f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad$ for $0 < |z - z_0| < dist(z_0, \partial\Omega)$.



$\underline{n = -1}$: $\quad a_n = \frac{1}{2\pi i} \oint_{C_r} f(\zeta) d\zeta = Res_{z = z_0} f(z)$.

Theorem.

1) z_0 is removable $\iff$ $\forall n<0,\ a_n=0$

2) z_0 is pole $\iff$ $\exists N\in\mathbb{N}:\ a_n=0\ \ \forall n<-N$.

3) z_0 is essential $\iff$ $\{n<0:\ a_n\neq 0\}$ is infinite.

Proof. 1) z_0 is removable $\iff f\in A(\Omega)$

So $a_{-n}=\frac{1}{2\pi i}\oint_{C_r} f(\zeta)(\zeta-z_0)^n\,d\zeta=0$, by Cauchy ($n>0$).

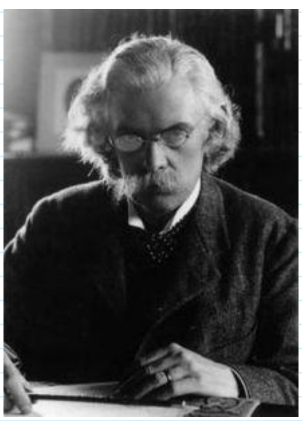
Other direction: $a_n=0\ \forall n<0$: $f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$ — analytic at z_0 ∎

2) already done: Theorem on characterization of poles.

3) Essential $\neq$ pole or removable

So $\{n<0:\ a_n\neq 0\}$ is infinite $\iff$ essential. ∎

Def. If z_0 is isolated singularity, $\sum_{n=1}^{\infty}\frac{a_{-n}}{(z-z_0)^n}$ is called singular part of Laurent decomposition.



Gösta Mittag-Leffler

Theorem (Mittag-Leffler).

Let $(z_n)\in\mathbb{C}$ be a sequence of distinct complex numbers, $z_n\to\infty$.

Let $g_n(z)=\sum_{k=1}^{l_n}\frac{a_{-k}^{(n)}}{(z-z_n)^k}$ — a sequence of polynomials in $\frac{1}{z-z_n}$.

Then there exists $f\in\mathcal{M}(\mathbb{C})$, with poles at just the points z_n and the corresponding singular parts $g_n(z)$.

Moreover, any meromorphic function in $\mathbb{C}$ has the form:

$f(z)=g(z)+\sum_{n=1}^{\infty}\left(g_n(z)-p_n(z)\right)$, where $g\in A(\mathbb{C})$ (entire),

$p_n(z)$ — a sequence of polynomials, $g_n(z)$ — singular parts at poles of f.

(partial fraction decomposition). Partial fraction decomposition for rational functions

Def. Let $f_n \in \mathcal{M}(\Omega)$. We say that $\sum f_n$ converges locally uniformly if $\forall K \subset \Omega$ - compact $\exists N(K): n \geq N(K) \Rightarrow f_n \in A(K)$, and $\sum_{n=N}^{\infty} f_n$ converges uniformly on K.

Remark. If $f = \sum f_n$, $f_n \in \mathcal{M}(\Omega)$ - locally uniformly, then $f \in \mathcal{M}(\Omega)$.

Proof. Need to prove: $\forall\ \overline{B(z,r)} \subset \Omega$, $f \in \mathcal{M}(\overline{B(z,r)})$. Take N from def.
But $f_0 := \sum_{n=N}^{\infty} f_n$ converges uniformly on $\overline{B(z,r)}$, so $f_0 \in A(\Omega)$.
So $f = f_0 + \underbrace{\sum_{n=1}^{N-1} f_n}_{\text{finite}} \in \mathcal{M}(\Omega)$. ∎

Proof (of Mittag-Leffler).

Observe: If $z_n \neq 0$, then $g_n(z) \in A(B(0,|z_n|))$ (it only has a singularity at z_n).
So for z: $|z| < \frac{|z_n|}{2}$, $g_n(z) = \sum b_k^{(n)} z^k$ - Taylor series, $b_k^{(n)} = \frac{g_n^{(k)}(0)}{k!}$.
Converges uniformly in $\overline{B(0, \frac{|z_n|}{2})}$. So $\exists\ m_n$:

$$\left| g_n(z) - \sum_{k=0}^{m_n} b_k^{(n)} z^k \right| < 2^{-n}.$$

Let $p_n(z) := \sum_{k=0}^{m_n} b_k^{(n)} z^k$.
Know: $\forall$ z with $|z| < \frac{|z_n|}{2}$, $|g_n(z) - p_n(z)| < 2^{-n}$.

Take K - compact. Since $z_n \to \infty$ $\exists N(K): \forall n \geq N(K)\ |z_n| \geq 2 \max_{z \in K} |z|$;
so $g_n \in A(K)$ and $|g_n(z) - p_n(z)| < 2^{-n}$.

Consider $f(z) := \sum_{z_n \neq 0} (g_n(z) - p_n(z)) + g_m(z)$, where $z_m = 0$
($g_m(z) = 0$ if no such z_m)

Then the series converges locally uniformly:
when $n \geq N(K)$, $g_n(z) - p_n(z) \in A(K)$, $|g_n(z) - p_n(z)| < 2^{-n}$
so $\sum_{n=N}^{\infty} (g_n(z) - p_n(z))$ - converges uniformly on K.

So $f \in \mathcal{M}(\mathbb{C})$. For $k \neq n$, $g_k(z) - p_k(z)$ is analytic at $B(z_n, r)$, $r = \min_{k \neq n} |z_n - z_k|$.
So the singular part of f at z_n is the same as the singular part of $g_n(z) - p_n(z)$, which is $g_n(z)$.

Finally, if f is a meromorphic function, let (z_n) be its poles,

$g_n(z)$ - singular parts. If (z_n) is finite - nothing to prove,
$f(z) - \sum g_n(z) \in A(\mathbb{C})$ - finite.
If (z_n) is infinite, $f(z) - \sum (g_n(z) - p_n(z)) \in A(\mathbb{C})$
as constructed

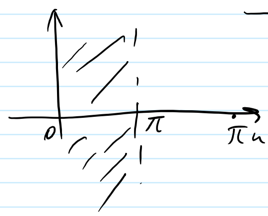
Examples 1) $f(z) = \frac{1}{\sin^2 z}$. Poles $z_n = \pi n$, of order 2.

$$g_n(z) = \frac{1}{(z-\pi n)^2} \qquad \left(\frac{1}{\sin^2 z} - \frac{1}{(z-\pi n)^2} = \frac{(z-\pi n)^2 - \sin^2 z}{\sin^2 z (z-\pi n)^2} \right)$$ expand in Taylor

$f_0(z) := \sum_{n=-\infty}^{\infty} \frac{1}{(z-\pi n)^2}$ - converges locally uniformly
(on each K, eventually $\frac{1}{(z-\pi n)^2} \le \frac{1}{n^2}$).

Observe: $f(z+\pi) = f(z)$, $f_0(z+\pi) = f_0(z)$,
so $h(z) := f(z) - f_0(z)$ - π-periodic: $h(z+\pi) = h(z)$.

Consider $0 \le \operatorname{Re} z \le \pi$. For $n \in \mathbb{N}$, $|z - n\pi| \ge \pi(n-1)$.
$|z + n\pi| \ge \pi n$.



$$\text{So } |f_0(z)| \le \frac{1}{|z|^2} + \sum_{n=-m}^{m} \frac{1}{|z-n\pi|^2} + 2\sum_{n=m+1}^{\infty} \frac{1}{(n-1)^2\pi^2}$$

$$\text{So } \varlimsup_{\substack{|z|\to\infty \\ 0 \le \operatorname{Re} z \le \pi}} |f_0(z)| \le 0 + 0 + 2\sum_{n=m+1}^{\infty} \frac{1}{\pi^2 (n-1)^2} \xrightarrow[m\to\infty]{} 0$$

$$\text{So } \lim_{\substack{|z|\to\infty \\ 0 \le \operatorname{Re} z \le \pi}} f_0(z) = 0.$$

Observe: ($z = x+iy$)
$$|\sin^2 z| = |\sin z|^2 = \left(\frac{e^{iz} - e^{-iz}}{2i}\right)\left(\frac{e^{-i\bar z} - e^{i\bar z}}{-2i}\right) =$$
$$\frac{1}{4}\left(-e^{2ix} - e^{-2ix} + e^{2y} + e^{-2y}\right) = \frac{1}{4}\left((e^{y} + e^{-y})^2 - (e^{ix} + e^{-ix})^2\right) =$$
$\cosh^2 y - \sin^2 x$.

As $z \to \infty$, $0 \le \operatorname{Re} z \le \pi$, we have $|y| \to \infty$, so
$\cosh^2 y \to \infty$, $|\sin x| \le 1$.

So $f(z) \to 0$.

So $h(z) \to 0$ as $|z| \to \infty$, $0 \le \operatorname{Re} z \le \pi$. So h is bounded in $0 \le \operatorname{Re} z \le \pi$.
But it is periodic, so $h \in A(\mathbb{C})$, bounded, so
$h \equiv$ const. Tends to 0, so $h \equiv 0$.

$$\text{So } \boxed{\frac{1}{\sin^2 z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z-\pi n)^2}}$$

$\cot z = \frac{\cos z}{\sin z}$ has simple poles at $z = n\pi$. Singular parts: $g_n(z) = \frac{1}{z - n\pi}$.
$\sum \frac{1}{z - n\pi}$ - diverges, but
$$\sum_{\substack{n=-\infty \\ n\ne 0}}^{\infty} \left(\frac{1}{z-n\pi} + \frac{1}{n\pi}\right) = \sum_{\substack{n=-\infty \\ n\ne 0}}^{\infty} \frac{z}{(z-n\pi)n} \text{ - converges.}$$
Can use similar technique, as above.

Another way:
Let γ be any arc from 0 to z, not passing through $\{\pi n\}$.

$$\oint_\gamma \left(\frac{1}{\sin^2\zeta} - \frac{1}{\zeta^2}\right) d\zeta = \frac{1}{z} - \cot z$$

$$\oint_\gamma \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{(\zeta - n\pi)^2} d\zeta \underset{\substack{\text{converges} \\ \text{uniformly on } \gamma}}{=} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \oint_\gamma \frac{d\zeta}{(\zeta-n\pi)^2} = -\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{1}{z-n\pi} + \frac{1}{n\pi}\right)$$

So

$$\boxed{\cot z - \frac{1}{z} = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{1}{z-n\pi} + \frac{1}{n\pi}\right)}$$

Theorem (Mittag-Leffler, local version)
Let $(z_n) \in \Omega$ be a sequence without limit points in Ω.
Let $g_n(z) = \sum_{k=1}^{\ell_n} \frac{a_{-k}^{(n)}}{(z-z_n)^k}$ – a sequence of polynomials of $\frac{1}{z-z_n}$.
Then there exists $f \in \mathcal{M}(\Omega)$, with poles at just the points z_n and the corresponding singular parts $g_n(z)$.
Moreover, any function in $\mathcal{M}(\Omega)$ has the form
$f(z) = g(z) + \sum_{n=1}^{\infty} (g_n(z) - p_n(z))$, where $g \in A(\Omega)$ (entire),
$p_n(z)$ – a sequence of polynomials, $g_n(z)$ – singular parts at poles of f.

No proof here.